# Derivation of low-temperature expansions for Ising model. V. Three-dimensional lattices-field grouping



1973 J. Phys. A: Math. Nucl. Gen. 6 1498

(http://iopscience.iop.org/0301-0015/6/10/008)

J. Phys. A: Math., Nucl. Gen., Vol. 6, October 1973. Printed in Great Britain. © 1973

# Derivation of low-temperature expansions for Ising model V. Three-dimensional lattices—field grouping

M F Sykes†, D S Gaunt†, J W Essam‡, B R Heap§, C J Elliott‖ and S R Mattingly†¶

† Wheatstone Physics Laboratory, King's College, London, UK
‡ Mathematics Department, Westfield College, London, UK
§ Division of Numerical Analysis and Computing, National Physical Laboratory, Teddington, UK
‖ Department of Physics, University of Alberta, Edmonton, Alberta, Canada

Received 10 May 1973

**Abstract.** A brief description is given of the derivation of series expansions for the three-dimensional Ising model of a ferromagnet and antiferromagnet as a high-field grouping. New results are given for the high-field polynomials for the face-centred cubic lattice to order 8, the body-centred cubic lattice to order 11, the simple cubic lattice to order 13 and the diamond lattice to order 17.

## 1. Introduction and summary

In this paper we extend the series expansions of four three-dimensional lattices, the face-centred cubic, body-centred cubic, simple cubic, and diamond, as a field or $\mu$ *grouping*. We have introduced the problem, defined the notation, and given the general theory in previous papers (Sykes *et al* 1965, 1973a,b,c to be referred to as I, II, III, IV respectively).

We give new results for the high-field polynomials $L_7$ and $L_8$ for the face-centred cubic lattice; for the complete code $F_5$ and $L_{10}$, $L_{11}$ for the body-centred cubic lattice; for $F_6$ and $L_{12}$, $L_{13}$ for the simple cubic lattice; for $F_7$, $F_8$ and $L_{14}$, $L_{15}$, $L_{16}$, $L_{17}$ for the diamond lattice.

The diamond lattice is of especial theoretical interest because it apparently yields low-temperature expansions for the spontaneous magnetization, and the specific heat and susceptibility in zero field, all of whose coefficients are of one sign (Block 1963, Essam and Sykes 1963). (This property is common to the white tin lattice (Block 1963) but it suffices to study the diamond lattice since it is clear from earlier work on the face-centred cubic and close-packed hexagonal lattices that the differences between these two systems will be extremely small (Domb and Sykes 1957).)

The extension of field groupings for the four lattices studied is a first logical step towards the extension of their temperature groupings which we describe subsequently (Sykes *et al* 1973d).

It is our main object to communicate new results; the length of the calculations makes it impractical to report them in detail. The actual mechanics of obtaining such

¶ Present Address: Department of Dynamical Climatology, Meteorological Office, London Road, Bracknell, UK.

data is rather specialized and rests often on ideas drawn from disciplines remote from physics and apparently of no relevance to the physical applications. The information contained in the complete codes has application to a wide variety of problems: they provide a concise summary of the sublattice polynomials (II § 2), for a study of order–disorder transitions in non-stoichiometric binary alloys (Bienenstock 1966, Bienenstock and Lewis 1967) and the staggered susceptibility, together with its field derivatives, of the Ising antiferromagnet (Rapaport and Domb 1971); they also arise in the derivation of low and high density expansions for hard sphere lattice gases (Gaunt and Fisher 1965, Gaunt 1967).

## 2. Derivation of complete codes for the diamond and face-centred cubic lattices

The diamond–face-centred cubic code system has already been introduced in II, § 3. It closely resembles that of the simple quadratic lattice. The four vertices of each shadow are vertices not of a square but of a tetrahedron and the shadow lattice is the face-centred cubic lattice. The shadow lattice is a first-neighbour lattice instead of a first- and second-neighbour lattice; but it is three dimensional instead of two dimensional and the enumeration is complicated by the necessity of distinguishing triangles and tetrahedra of significant and insignificant parity. On balance the problem is a little easier, the number of graphical codes at each order being slightly less than for the simple quadratic system.

As we have shown in II, § 3, the partial generating functions or complete codes for the diamond lattice can also be regarded as codes for the face-centred cubic lattice; for the former the appropriate substitutions are given by (2.6) and (2.7) of II and for the latter by (3.9) of II.

We formalize the concept of the diamond–face-centred cubic code system by a straightforward generalization of the treatment of III, § 2. If $n$ tetrahedral shadows do not touch they correspond to the code $(4n, 4n)$. In general any strong embedding, in the face-centred cubic lattice, of $n$ sites forming $T$ tetrahedra of significant parity, $t$ triangles of significant parity (which do not lie in tetrahedra), and $p$ edges which do not lie in tetrahedra or triangles of significant parity (and therefore correspond to pair contacts) yields a code:

$$(4n-p-2t-3T, 4n-2p-3t-4T, p, t, T) \tag{2.1}$$

and this defines the *algebraic* code system. The code system is identical in form with the simple quadratic system (3.1) of III, but differs in the interpretation of $p$ on the shadow lattice. Not every code in (2.1) occurs on the lattice. For example, from (2.1) by setting $T = t = 0$ and $n = 8$ we obtain one possible sequence of codes of eighth order:

$$(32, 32), (31, 30, 1), \ldots, (19, 6, 13), (18, 4, 14), (17, 2, 15), (16, 0, 16). \tag{2.2}$$

It is possible to pile eight tetrahedral shadows so as to have 13 contacts between pairs (as we illustrate in figure 1); it is not possible to find arrangements with more than 13 such contacts: the last three codes of the sequence (2.2) are non-graphical for the diamond–face-centred cubic code system. (The fact that a code is non-graphical for the diamond–face-centred cubic code system does not necessarily imply that it is non-graphical for the white-tin–closed-packed hexagonal system; the latter system has the same algebraic codes defined by (2.1).) The total number of graphical codes in $F_8$ for the diamond lattice is 91.

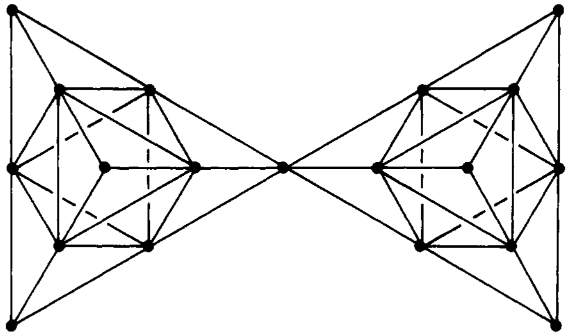

**Figure 1.** Eight tetrahedral shadows arranged to correspond to the code (19,6,13) on the diamond lattice. There are 6 isolated vertices and 13 contacts between pairs.

To obtain the complete codes $F_7$ and $F_8$ we have followed III, § 3 and counted and coded all seven and eight point graphs; as before we have exploited the principle of complete code balance to find the contribution of some separated graphs. The complete list of eight point graphs was generated on the KDF9 computer at the National Physical Laboratory using methods developed by one of us (BRH). Much of the counting and coding was done by computer using a program developed for this purpose (Elliott 1969). We give the codes $F_7$ and $F_8$ together with the high-field polynomials $L_{14}$, $L_{15}$, 1969). We give in appendixes the codes $F_7$ and $F_8$ together with the high-field polynomials $L_{14}$, $L_{15}$, $L_{16}$ and $L_{17}$ for the diamond lattice derived from them.

$$(\lambda, \alpha, \beta, \gamma, \delta) = u^{\frac{1}{2}(3\alpha+4\beta+3\gamma)}\mu^{\frac{1}{2}(\alpha+2\beta+3\gamma+4\delta)} \tag{2.3}$$

we obtain for the face-centred cubic lattice the high-field polynomials:

$$\begin{aligned} L_7 = {} & 8u^{27}+36u^{28}+336u^{29}+1350u^{30}+3528u^{31}+9036u^{32}-1160u^{33}+1038u^{34} \\ & -281400u^{35}-622498u^{36}+1503912u^{37}+8356041u^{38}-28260664u^{39} \\ & +34148478u^{40}-18902160u^{41}+4044119\tfrac{1}{7}u^{42} \end{aligned} \tag{2.4}$$

$$\begin{aligned} L_8 = {} & 28u^{30}+96u^{31}+786u^{32}+2432u^{33}+9804u^{34}+19314u^{35}+29146u^{36} \\ & +20550u^{37}-322950u^{38}-474806u^{39}-4371355\tfrac{1}{2}u^{40}+1944846u^{41} \\ & +40271875u^{42}+32438508u^{43}-452857765\tfrac{1}{2}u^{44}+916579240u^{45} \\ & -853695741u^{46}+393105420u^{47}-72699427\tfrac{1}{8}u^{48}. \end{aligned} \tag{2.5}$$

## 3. Derivation of complete codes for the simple cubic and body-centred cubic lattices

The code system for the simple cubic lattice corresponds to the possible contacts between octahedral shadows. The piling of octahedra together is visually difficult; the corresponding shadow lattice is the face-centred cubic with first and second neighbours. If for any configuration of shadows we denote by $n_s$ the number of points common to $s$ shadows, the algebraic code system can be written:

$$(6n-n_2-2n_3-3n_4-4n_5-5n_6,\ 6n-2n_2-3n_3-4n_4-5n_5-6n_6,\ n_2, n_3, n_4, n_5, n_6). \tag{3.1}$$

The code system for the body-centred cubic lattice corresponds to the possible contacts between cubic shadows. The piling of cubes is visually simpler but the number of different types of contact is larger; the corresponding shadow lattice is the simple cubic lattice with first, second and third neighbours. The maximum possible value of $s$ is now eight and the algebraic code system can be written:

$$(8n-n_2-2n_3-3n_4-4n_5-5n_6-6n_7-7n_8,\, 8n-2n_2-3n_3-4n_4-5n_5-6n_6 \\ -7n_7-8n_8,\, n_2,\, n_3,\, n_4,\, n_5,\, n_6,\, n_7,\, n_8). \tag{3.2}$$

In both (3.1) and (3.2) the quantities $n_s$ can be given graphical interpretations on the shadow lattice but we have generally not found these useful. Because of the complexity of the code systems it is more difficult to exploit the principle of complete code balance to find the contributions of separated graphs. We have derived by direct counting the complete code $F_6$ for the simple cubic lattice, and the complete code $F_5$ for the body-centred cubic lattice. We give these, together with the derived high-field polynomials, in appendixes.

## Acknowledgments

This research has been supported (in part) by a grant from the Science Research Council and (in part) by the US Department of the Army through its European Office.

## Appendix 1. Partial generating functions

Body-centred cubic lattice

$$\begin{aligned}
F_5 = {} & 24(22, 11, 7, 2, 1, 1)+24(22, 10, 8, 2, 2)+24(23, 12, 7, 3, 0, 1)+12(24, 16, 2, 4, 2) \\
& +3(24, 16, 0, 8)+48(24, 15, 4, 3, 2)+8(24, 14, 6, 3, 0, 1)+48(24, 14, 6, 2, 2) \\
& +96(24, 14, 5, 4, 1)+36(24, 14, 4, 6)+48(24, 13, 8, 1, 2)+24(24, 13, 7, 3, 1) \\
& +12(24, 12, 10, 0, 2)+48(24, 12, 9, 2, 1)+108(24, 12, 8, 4)+36(24, 10, 12, 2) \\
& +3(24, 8, 16)+24(25, 16, 5, 2, 2)+48(25, 15, 7, 1, 2)+96(25, 15, 6, 3, 1) \\
& +24(25, 14, 9, 0, 2)+144(25, 14, 8, 2, 1)+48(25, 13, 10, 1, 1) \\
& +48(26, 18, 3, 4, 1)+24(26, 18, 2, 6)+336(26, 17, 5, 3, 1)+144(26, 17, 4, 5) \\
& -132(26, 16, 8, 0, 2)+336(26, 16, 7, 2, 1)+408(26, 16, 6, 4) \\
& +24(26, 15, 9, 1, 1)+648(26, 15, 8, 3)+516(26, 14, 10, 2)+120(26, 13, 12, 1) \\
& +72(26, 12, 14)+104(27, 19, 4, 3, 1)+24(27, 19, 3, 5)+480(27, 18, 6, 2, 1) \\
& +432(27, 18, 5, 4)+336(27, 17, 8, 1, 1)+168(27, 17, 7, 3) \\
& +48(27, 16, 10, 0, 1)+744(27, 16, 9, 2)+48(27, 15, 11, 1)+96(27, 14, 13) \\
& -432(28, 21, 3, 3, 1)-1296(28, 20, 5, 2, 1)-984(28, 20, 4, 4) \\
& +336(28, 19, 7, 1, 1)+1752(28, 19, 6, 3)+72(28, 18, 9, 0, 1) \\
& -1008(28, 18, 8, 2)+1464(28, 17, 10, 1)-36(28, 16, 12) \\
& -1344(29, 21, 6, 1, 1)+1752(29, 21, 5, 3)-312(29, 20, 8, 0, 1)
\end{aligned}$$

$+3912(29,20,7,2)+2616(29,19,9,1)+1656(29,18,11)$
$-5664(30,23,4,3)+32(30,22,7,0,1)-8820(30,22,6,2)$
$-4728(30,21,8,1)-5340(30,20,10)-120(31,24,6,0,1)$
$-12504(31,24,5,2)+6168(31,23,7,1)-2016(31,22,9)$
$+720(32,24,8)-24480(32,25,6,1)+16668(32,26,4,2)$
$-56720(33,26,7)-39144(33,27,5,1)+80604(34,28,6)$
$+43968(34,29,4,1)+18240(35,30,5)+19440(35,31,3,1)$
$-50248(36,32,4)+346008(37,34,3)-281208(38,36,2)$
$-334864(39,38,1)+279850\tfrac{1}{5}(40,40).$

Simple cubic lattice

$F_6 = 1(19,6,12,0,0,0,1)+24(21,10,9,1,0,1)+24(21,12,5,3,0,1)$
$+24(22,11,10,0,0,1)+56(22,12,6,4)+144(22,12,7,2,1)$
$+48(22,12,8,0,2)+96(22,12,8,1,0,1)+24(22,13,5,3,1)$
$+48(22,14,3,4,1)+96(22,14,4,2,2)+6(22,16,0,4,2)$
$+12(23,12,10,0,1)+264(23,13,7,3)+456(23,13,8,1,1)$
$+54(23,13,9,0,0,1)+288(23,14,5,4)+720(23,14,6,2,1)$
$+60(23,14,7,0,2)+72(23,15,3,5)+360(23,15,4,3,1)$
$+96(23,16,2,4,1)+4(24,12,12)+1632(24,14,8,2)+780(24,14,9,0,1)$
$+2472(24,15,6,3)+2328(24,15,7,1,1)-258(24,15,8,0,0,1)$
$+1020(24,16,4,4)+816(24,16,5,2,1)+6(24,16,6,0,2)+96(24,17,2,5)$
$+336(24,17,3,3,1)+6(25,14,11)+5136(25,15,9,1)+11340(25,16,7,2)$
$+2232(25,16,8,0,1)+4560(25,17,5,3)-12(25,17,6,1,1)$
$+1120(25,18,3,4)+84(25,18,4,2,1)-1092(25,19,2,3,1)$
$+5976(26,16,10)+28152(26,17,8,1)+11578(26,18,6,2)$
$-7044(26,18,7,0,1)-2016(26,19,4,3)-6672(26,19,5,1,1)$
$-2064(26,20,2,4)+33798(27,18,9)-16380(27,19,7,1)$
$-51612(27,20,5,2)-10422(27,20,6,0,1)-8052(27,21,3,3)$
$-1224(27,21,4,1,1)-1578(27,22,1,4)-72288(28,20,8)$
$-248496(28,21,6,1)-64020(28,22,4,2)+13581(28,22,5,0,1)$
$-1980(28,23,2,3)+2519(28,24,0,4)-507354(29,22,7)$
$+47856(29,24,3,2)+4332(29,24,4,0,1)-70140(29,23,5,1)$
$+341606(30,24,6)+614112(30,25,4,1)+40176(30,26,2,2)$
$+2418105(31,26,5)+148636(31,27,3,1)-1193821\tfrac{1}{2}(32,28,4)$

$-446340(32, 29, 2, 1) - 4395996(33, 30, 3) + 2392518(34, 32, 2)$
$+ 2610540(35, 34, 1) - 1641564\tfrac{2}{3}(36, 36).$

Diamond lattice

$$
\begin{aligned}
F_7 = {} & 6(16, 6, 8, 2) + 12(17, 7, 9, 1) + 24(17, 8, 7, 2) + 4(17, 10, 3, 4) + 72(18, 8, 10) \\
& + 480(18, 9, 8, 1) + 498(18, 10, 6, 2) + 36(18, 10, 7, 0, 1) + 96(18, 11, 4, 3) \\
& + 24(18, 11, 5, 1, 1) + 1832(19, 10, 9) + 2736(19, 11, 7, 1) + 1428(19, 12, 5, 2) \\
& + 160(19, 12, 6, 0, 1) + 208(19, 13, 3, 3) + 132(19, 13, 4, 1, 1) + 24(19, 14, 1, 4) \\
& + 4(19, 15, 0, 3, 1) - 1482(20, 12, 8) + 7248(20, 13, 6, 1) + 6240(20, 14, 4, 2) \\
& + 792(20, 14, 5, 0, 1) + 816(20, 15, 2, 3) + 492(20, 15, 3, 1, 1) - 32(20, 16, 0, 4) \\
& + 24(20, 16, 1, 2, 1) + 6(20, 16, 2, 0, 2) + 82548(21, 14, 7) + 44052(21, 15, 5, 1) \\
& + 168(21, 16, 3, 2) + 744(21, 16, 4, 0, 1) + 96(21, 17, 1, 3) + 12(21, 17, 2, 1, 1) \\
& + 12(21, 18, 0, 2, 1) - 636948(22, 16, 6) - 389172(22, 17, 4, 1) \\
& - 40116(22, 18, 2, 2) - 9608(22, 18, 3, 0, 1) - 604(22, 19, 0, 3) \\
& - 852(22, 19, 1, 1, 1) - 455352(23, 18, 5) - 2128(23, 19, 3, 1) \\
& + 1428(23, 20, 1, 2) - 1416(23, 20, 2, 0, 1) - 196(23, 21, 0, 1, 1) \\
& + 10941345(24, 20, 4) + 1959264(24, 21, 2, 1) + 30766(24, 22, 0, 2) \\
& + 23796(24, 22, 1, 0, 1) - 29221216(25, 22, 3) - 2585304(25, 23, 1, 1) \\
& - 14188(25, 24, 0, 0, 1) + 34148478(26, 24, 2) + 960552(26, 25, 0, 1) \\
& - 18902160(27, 26, 1) + 4044119\tfrac{1}{7}(28, 28).
\end{aligned}
$$

$$
\begin{aligned}
F_8 = {} & 12(18, 6, 10, 2) + 6(18, 8, 6, 4) + 6(19, 6, 13) + 12(19, 7, 11, 1) + 96(19, 8, 9, 2) \\
& + 3(19, 8, 10, 0, 1) + 84(19, 9, 7, 3) + 12(19, 9, 8, 1, 1) - 24\tfrac{1}{2}(20, 8, 12) \\
& + 372(20, 9, 10, 1) + 888(20, 10, 8, 2) + 48(20, 10, 9, 0, 1) + 564(20, 11, 6, 3) \\
& + 180(20, 11, 7, 1, 1) + 114(20, 12, 4, 4) + 24(20, 12, 5, 2, 1) + 24(20, 13, 2, 5) \\
& + 16(20, 13, 3, 3, 1) + 2106(21, 10, 11) + 8820(21, 11, 9, 1) + 9576(21, 12, 7, 2) \\
& + 702(21, 12, 8, 0, 1) + 3600(21, 13, 5, 3) + 1032(21, 13, 6, 1, 1) \\
& + 136(21, 14, 3, 4) + 210(21, 14, 4, 2, 1) + 12(21, 14, 5, 0, 2) \\
& + 24(21, 15, 2, 3, 1) + 17475(22, 12, 10) + 26952(22, 13, 8, 1) \\
& + 17064(22, 14, 6, 2) + 1842(22, 14, 7, 0, 1) + 5076(22, 15, 4, 3) \\
& + 3012(22, 15, 5, 1, 1) + 774(22, 16, 2, 4) + 612(22, 16, 3, 2, 1) \\
& + 87(22, 16, 4, 0, 2) + 12(22, 17, 0, 5) + 84(22, 17, 1, 3, 1) + 6(22, 18, 0, 2, 2) \\
& - 29388(23, 14, 9) + 54876(23, 15, 7, 1) + 61938(23, 16, 5, 2) \\
& + 7278(23, 16, 6, 0, 1) + 12728(23, 17, 3, 3) + 3924(23, 17, 4, 1, 1) \\
& - 756(23, 18, 1, 4) + 1044(23, 18, 2, 2, 1) + 84(23, 18, 3, 0, 2)
\end{aligned}
$$

$$-152(23,19,0,3,1)+12(23,19,1,1,2)+642427\tfrac{1}{2}(24,16,8)$$
$$+41324(24,17,6,1)-235080(24,18,4,2)-24558(24,18,5,0,1)$$
$$-27588(24,19,2,3)-17064(24,19,3,1,1)+888(24,20,0,4)$$
$$-732(24,20,1,2,1)-228(24,20,2,0,2)-9630870(25,18,7)$$
$$-6421896(25,19,5,1)-758826(25,20,3,2)-151593(25,20,4,0,1)$$
$$-24756(25,21,1,3)-18300(25,21,2,1,1)-516(25,22,0,2,1)$$
$$+6(25,22,1,0,2)+16993932(26,20,6)+11444448(26,21,4,1)$$
$$+1071264(26,22,2,2)+247004(26,22,3,0,1)+13828(26,23,0,3)$$
$$+16128(26,23,1,1,1)-27\tfrac{1}{2}(26,24,0,0,2)+110567592(27,22,5)$$
$$+23960992(27,23,3,1)+856704(27,24,1,2)+336849(27,24,2,0,1)$$
$$+11252(27,25,0,1,1)-522167473\tfrac{1}{2}(28,24,4)-78129084(28,25,2,1)$$
$$-990252(28,26,0,2)-725436(28,26,1,0,1)+936899592(29,26,3)$$
$$+69309708(29,27,1,1)+307203(29,28,0,0,1)-853695741(30,28,2)$$
$$-20320352(30,29,0,1)+393105420(31,30,1)-72699427\tfrac{1}{8}(32,32).$$

## Appendix 2. High-field polynomials $L(u)$

Body-centred cubic lattice

$$\begin{aligned}L_{10} = {} & 156u^{23}+2418u^{24}+19568u^{25}+89832u^{26}+312984u^{27}+534960u^{28} \\ & -582528u^{29}-21524820u^{30}-122555960u^{31}+184704162u^{32} \\ & +4891550184u^{33}-25940728064u^{34}+62669293900\tfrac{4}{5}u^{35} \\ & -88827538116u^{36}+78607759128u^{37}-42991931004u^{38} \\ & +13362730248u^{39}-1812137048\tfrac{9}{10}u^{40}.\end{aligned}$$

$$\begin{aligned}L_{11} = {} & 12u^{24}+800u^{25}+9720u^{26}+65112u^{27}+302497u^{28}+897848u^{29} \\ & +1976484u^{30}-2366032u^{31}-34701994u^{32}-284193600u^{33} \\ & -704476488u^{34}+6025344368u^{35}+36918882951u^{36} \\ & -323871127432u^{37}+1029543128536u^{38}-1871827463448u^{39} \\ & +2164621975492u^{40}-1630783111424u^{41}+779883805680u^{42} \\ & -215938102896u^{43}+26449153814\tfrac{1}{11}u^{44}.\end{aligned}$$

Simple cubic lattice

$$\begin{aligned}L_{12} = {} & 3u^{16}+1080u^{18}+11562u^{19}+101685u^{20}+814709u^{21}+3894597u^{22} \\ & -12171177u^{23}-135740953u^{24}-397387542u^{25}+4338189541\tfrac{1}{2}u^{26} \\ & +11093270424\tfrac{2}{3}u^{27}-170115111953\tfrac{1}{4}u^{28}+682270008351u^{29}\end{aligned}$$

$$-1542754484221u^{30}+2260372621941u^{31}-2238908395410u^{32}$$
$$+1498634619771u^{33}-652575075531u^{34}+167442968667u^{35}$$
$$-19258135545u^{36}.$$

$$L_{13} = 96u^{18}+732u^{19}+23976u^{20}+163820u^{21}+1256172u^{22}+6874170u^{23}$$
$$+12343160u^{24}-220608330u^{25}-1032194100u^{26}+2269586155u^{27}$$
$$+43210929384u^{28}-18514105314u^{29}-1306808581968u^{30}$$
$$+7163363995983u^{31}-20147356102164u^{32}+36242844825794u^{33}$$
$$-44637329262900u^{34}+38365757618721u^{35}-22758644334336u^{36}$$
$$+8917222503222u^{37}-2082822677172u^{38}+220080372439\tfrac{1}{13}u^{39}$$

Diamond lattice

$$L_{14} = 2u^{10}+240u^{11}+18836u^{12}+515810u^{13}+7916204u^{14}-58478578u^{15}$$
$$-1418314198u^{16}+21334110908u^{17}-138980195662u^{18}$$
$$+555924571090u^{19}-1521665590807\tfrac{1}{2}u^{20}+2991146529116\tfrac{2}{7}u^{21}$$
$$-4315381895540u^{22}+4592751408686u^{23}-3572264632004u^{24}$$
$$+1977785317192u^{25}-739153000732u^{26}+167308649600u^{27}$$
$$-17336930162\tfrac{6}{7}u^{28}.$$

$$L_{15} = 40u^{11}+3166u^{12}+154248u^{13}+3199836u^{14}+32944408u^{15}-476674482u^{16}$$
$$-5773707720u^{17}+118886540506\tfrac{1}{3}u^{18}-900637157640u^{19}$$
$$+4097550151113\tfrac{1}{5}u^{20}-12741101868598\tfrac{2}{3}u^{21}+28650886947846u^{22}$$
$$-47891909381080u^{23}+60175860118355\tfrac{1}{3}u^{24}-56771186125459\tfrac{1}{5}u^{25}$$
$$+39681684986502u^{26}-19960390222954\tfrac{2}{3}u^{27}+6838587129336u^{28}$$
$$-1429734380020u^{29}+137717342597\tfrac{11}{15}u^{30}.$$

$$L_{16} = 626u^{12}+33634u^{13}+1148682u^{14}+18106680u^{15}+110230585\tfrac{1}{2}u^{16}$$
$$-3234261150u^{17}-19541105053u^{18}+640954742858u^{19}$$
$$-5685110554290\tfrac{1}{2}u^{20}+29306941485862u^{21}-102700495543167u^{22}$$
$$+261215271993786u^{23}-498289778476257\tfrac{3}{4}u^{24}+724299694749954u^{25}$$
$$-805866713062893u^{26}+682656831924498u^{27}-433224056104135\tfrac{1}{2}u^{28}$$
$$+199652521843564u^{29}-63145125857905u^{30}+12265456611610u^{31}$$
$$-1103747907487\tfrac{13}{16}u^{32}.$$

$$L_{17} = 42u^{12}+7920u^{13}+308249u^{14}+7844420u^{15}+92790088u^{16}+180320296u^{17}$$
$$-19705003862u^{18}-37115043752u^{19}+3324202355814u^{20}$$
$$-34966832701788u^{21}+203911552422240u^{22}-800560594465168u^{23}$$
$$+2282954119079018u^{24}-4911907187828912u^{25}+8133709804858208u^{26}$$

$$-10456683268746776u^{27}+10437353943227692u^{28}$$
$$-8021829992619180u^{29}+4661326193944755u^{30}-1982031993840852u^{31}$$
$$+582129436627018u^{32}-105588404881920u^{33}+891556134645\tfrac{1}{17}u^{34}.$$

## References


Bienenstock A 1966 *J. appl. Phys.* **37** 1459–61
Bienenstock A and Lewis J 1967 *Phys. Rev.* **160** 393–403
Block E 1963 *Ark. Fys.* **24** 79–80
Domb C and Sykes M F 1957 *Proc. Phys. Soc.* B **70** 896–7
Elliott C J *PhD Thesis* University of Alberta
Essam J W and Sykes M F 1963 *Physica* **29** 378–88
Gaunt D S 1967 *J. chem. Phys.* **46** 3237–59
Gaunt D S and Fisher M E 1965 *J. chem. Phys.* **43** 2840–63
Rapaport D C and Domb C 1971 *J. Phys.* C: *Solid St. Phys.* **4** 2684–94
Sykes M F, Essam J W and Gaunt D S 1965 *J. math. Phys.* **6** 283–98
Sykes M F, Gaunt D S, Essam J W and Hunter D L 1973a *J. Math. Phys.* **14** 1060–5
Sykes M F *et al* 1973b *J. math. Phys.* **14** 1066–70
Sykes M F *et al* 1973c *J. math. Phys.* **14** 1071–4
Sykes M F *et al* 1973d *J. Phys.* A: *Math., Nucl. Gen.* **6** 1507–16